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THE SOLUTION OF THE EQUATIONS OF LINEAR ELASTICITY FOR AN INFINITE REGION CONTAINING TWO SPHERICAL INCLUSIONS

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Abstract—In this paper, exact expressions for the stresses and the displacements in an infinite elastic solid containing two spherical inclusions are presented when an arbitrary linear strain field is applied at infinity. Owing to the linearity of the elasticity problem, the general solution can be obtained by superposing the stresses and displacements that result from the application of four independent strains at infinity. Two of these cases lead to axisymmetric solutions which are evaluated for elastic particles, while the remaining are solved only for rigid inclusions and cavities.

The analysis is based on the Boussinesq–Papkovich stress function approach and makes use of the "multipole expansion" technique in which the solutions are expanded into series of spherical harmonics with respect to the centers of the two spheres. The solutions thus obtained converge very rapidly when the spheres are more than three radii apart, but become slowly convergent as the separation decreases.

Numerical results are presented in graphs for the stresses along the center-line between two cavities and between two rigid spheres. In the latter case, the displacements of the rigid particles are also calculated.

1. INTRODUCTION

The equilibrium problem in the theory of elasticity for a region containing two spherical inclusions of the same size is of technical interest because its solution demonstrates the interference between two sources of stress concentration. Since, as will be seen below, the complete analysis for an arbitrary strain field applied at infinity is rather complicated, previous studies have been restricted to the axisymmetric problems for an infinite region containing either cavities or rigid particles [1–4]. The only exception appears to be a recent paper by Tsuchida *et al.* [5] who solved the problem when the applied field consists of an uniaxial tension in the direction perpendicular to the line of centers of the cavities.

Two standard methods have been developed for treating two-sphere problems. The first is in terms of bispherical (spherical bipolar) coordinates and was used by Sternberg and Sadowsky [1] for cavities and by Shelley and Yu [2] for rigid spheres under hydrostatic tension or under an uniaxial tension along the line of centers of the inclusions. The second, employed by Miyamoto [3,4] for cavities under uniaxial tension along their line of centers and recently by Tsuchida *et al.* [5] in the article referenced above, is the "multipole expansion" technique in which the solutions are expanded into series of spherical harmonics with respect to the centers of both spheres. A comparison of these two methods shows that the former requires the numerical solution of a set of infinite linear equations for each separation distance between the spheres which, in the past [1,2], was accomplished by truncating the infinite set of equations. In contrast, the latter method requires the derivation of recurrence formulae for relating the coefficients of the spherical harmonics. Although such an approach also leads to an infinite series, the solution can be expressed explicitly as a function of the separation distance and can, in principle, be evaluated numerically to any desired degree of accuracy by retaining the appropriate number of terms. Consequently, this second method appeared to be more suitable for treating the present problem.

In this paper, we shall generalize the earlier results referred to above and shall consider the problem of an infinite elastic solid containing two spherical inclusions of the same size in the presence of an arbitrary but constant applied strain at infinity. Owing to the linearity of the problem, the solution for this general case can be obtained by superimposing the solutions for the

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following four independent applied strains:

$$(a)\ \epsilon_{ij} = \delta_{ij}, \quad (b)\ \epsilon_{ij} = \delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2} - 2\delta_{i3}\delta_{j3},$$
$$(c)\ \epsilon_{ij} = \delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1}, \quad (d)\ \epsilon_{ij} = \delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2}, \tag{1a–d}$$

where the x_3-axis is along the line of centers of the spheres and δ_{ij} denotes the Kronecker delta. Of the four strains, the first two result in axisymmetric problems which will be solved for the general case of two elastic spheres in an infinite region. However, (c) and (d) are more complicated and their solutions require a great deal of effort. We shall, therefore, examine in detail the two limiting cases in which the spheres are either rigid particles or cavities. The method of solution will be presented for these applied strains in Sections 3–6.

As shown in a paper by Chen and Acrivos[6] and in Chen's thesis[7], the solutions of the above problems are needed for the determination of the bulk stresses in a composite material containing spherical inclusions in sufficiently large concentrations for particle–particle interactions to be important. This paper is thus intended to provide information about the interaction between two spheres embedded in an infinite domain, with a view to its use in [6] for the evaluation of the stresslet S_{ij}, defined by

$$S_{ij} = \int_{S_p} (x_j\sigma_{ik}n_k - \lambda_o u_k n_k \delta_{ij} - \mu_o(u_i n_j + u_j n_i))\,\mathrm{d}S,$$

where S_p denotes the surface of the particle in question and n_i is its unit outer normal, and for the calculation of the effective elastic moduli of such a composite material. We shall now proceed with the outline of the method of solution.

2. STRESS FUNCTIONS AND METHOD OF SOLUTION

The coordinates of the two-sphere system are illustrated in Fig. 1. The spheres are both of radius a and their centers are located at $O(x,y,z)$ and $O_1(x_1,y_1,z_1)$, respectively. zz_1 is along the line of centers of the two spheres and the distance between the centers, OO_1, is denoted by R.

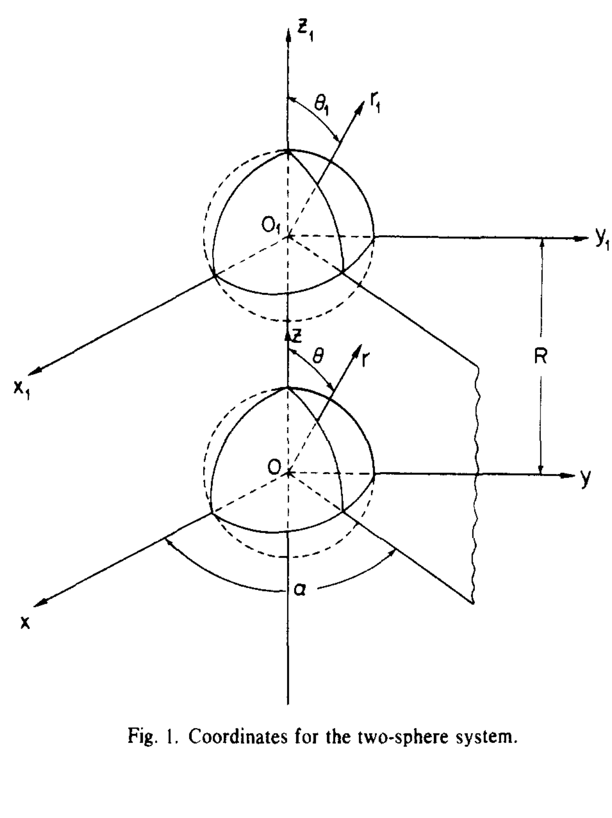

Fig. 1. Coordinates for the two-sphere system.

The two sets of coordinates, one for each sphere, are related by

$$x = x_1, \quad y = y_1, \quad z = z_1 + R.$$

In terms of the spherical coordinates, we have

$$\begin{aligned} x &= r\sin\theta\cos\alpha & x_1 &= r_1\sin\theta_1\cos\alpha \\ y &= r\sin\theta\sin\alpha & y_1 &= r_1\sin\theta_1\sin\alpha \\ z &= r\cos\theta & z_1 &= r_1\cos\theta_1 \end{aligned}$$

where $0 \le r, r_1 \le \infty, 0 \le \theta, \theta_1 \le \pi, 0 \le \alpha \le 2\pi$.

The general solution for the displacement equations in the absence of body forces,

$$\frac{\partial^2 u_j}{\partial x_i \partial x_j} + (1-2\nu)\frac{\partial^2 u_i}{\partial x_j \partial x_j} = 0 \tag{2}$$

can be represented using the Boussinesq–Papkovich stress functions [8] as

$$\begin{aligned} 2\mu\, u_i &= \frac{\partial \psi}{\partial x_i} + x_j\frac{\partial \tau_j}{\partial x_i} - (3-4\nu)\tau_i \\ &= \frac{\partial \psi}{\partial x_i} + \frac{\partial (x_j\tau_j)}{\partial x_i} - 4(1-\nu)\tau_i \end{aligned} \tag{3}$$

where $\nabla^2\psi = \nabla^2\tau_i = 0$ and ν is Poisson's ratio of the medium. For the two-sphere system, κ_o, μ_o, ν_o and κ_p , μ_p, ν_p will denote the bulk modulus, shear modulus and Poisson's ratio of the surrounding medium and of the particles, respectively. Once u_i has been obtained, the associated stress fields can then be determined from Hooke's law.

It is well known[9], that, of the four harmonic functions given in (3), only three are independent. The advantage of retaining all of them is that we can arbitrarily either eliminate one or combine two of them to handle particular cases. When the problem is axisymmetric, only two functions are needed[8], namely ψ and τ_3, with x_3 being the axis of symmetry.

The boundary conditions that have to be satisfied are that, as $|x_i| \to \infty$, the displacement field has to approach that of the corresponding loading and that the displacement and the traction be continuous on the surface of each inclusion. In addition, for the case of rigid particles, whose displacement and rotation need to be determined as part of the solution, we require that the total force and couple acting on each inclusion be zero.

We shall now proceed with the solution to our problem for the four applied strains, (1a–d).

3. AXISYMMETRIC SOLUTION FOR $\epsilon_{ij}^{\infty} = \delta_{ij}$

With the strain given at infinity as $\epsilon_{ij}^{\infty} = \delta_{ij}$, the corresponding displacement and stress fields are

$$u_i^{\infty} = x_i^*$$

and

$$\sigma_{ij}^{\infty} = 3\kappa_0\delta_{ij},$$

where x_i^* is taken from the midpoint of the two centers of the spheres. In spherical coordinates with origin at O, the center of one of the spheres (see Fig. 1), the above become

$$u_r^{\infty} = r - \frac{R}{2}\cos\theta, \quad u_\theta^{\infty} = \frac{R}{2}\sin\theta, \quad u_\alpha^{\infty} = 0, \tag{4a}$$

and

$$\sigma_{rr}^{\infty} = \sigma_{\theta\theta}^{\infty} = \sigma_{\alpha\alpha}^{\infty} = 3\kappa_o, \quad \sigma_{r\theta}^{\infty} = \sigma_{r\alpha}^{\infty} = \sigma_{\theta\alpha}^{\infty} = 0. \tag{4b}$$

As mentioned in the previous section, the solution to this axisymmetric system will involve only two stress functions ψ and τ_3 which depend only on r and θ. The displacement field can therefore be represented as:

(a) *Outside the spheres*

$$2\mu_0\mathbf{u} = 2\mu_0\mathbf{u}^\infty + \text{grad}\,\zeta + \text{grad}\,(z\psi + z_1\psi_1) - 4(1-\nu_0)(\psi + \psi_1)\mathbf{e}_z \tag{5}$$

$$\zeta = \sum_{n=0}^{\infty}\left\{A_n \frac{a^{n+3}}{r^{n+1}} P_n(\cos\theta) + A_n{}^1 \frac{a^{n+3}}{r_1^{n+1}} P_n(\cos\theta_1)\right\}$$

$$\psi = \sum_{n=0}^{\infty} C_n \frac{a^{n+2}}{r^{n+1}} P_n(\cos\theta), \quad \psi_1 = \sum_{n=0}^{\infty} C_n{}^1 \frac{a^{n+2}}{r_1^{n+1}} P_n(\cos\theta_1),$$

where $P_n(\cos\theta)$ is the Legendre Polynomial of order n with argument $\cos\theta$ and a is the radius of the spheres.

(b) *Inside sphere O*

$$2\mu_p\mathbf{u} = \text{grad}\,\eta + \text{grad}(z\xi) - 4(1-\nu_p)\xi\mathbf{e}_z \tag{6}$$

where

$$\eta = \sum_{n=0}^{\infty} B_n \frac{r^n}{a^{n-2}} P_n(\cos\theta); \qquad \xi = \sum_{n=0}^{\infty} D_n \frac{r^n}{a^{n-1}} P_n(\cos\theta).$$

An expression similar to (6) applies for the region inside sphere O_1. The relations between the harmonic functions referred to each of the two spheres are given by Hobson[10],

$$\frac{P_n{}^m(\cos\theta)}{r^{n+1}} = \sum_{s=m}^{\infty} \frac{(s+n)!}{(s+m)!(n-m)!} \frac{r_1{}^s}{R^{s+n+1}} (-1)^{s+m} P_s{}^m(\cos\theta_1) \tag{7a}$$

and

$$\frac{P_n{}^m(\cos\theta_1)}{r_1^{n+1}} = \sum_{s=m}^{\infty} \frac{(s+n)!}{(s+m)!(n-m)!} \frac{r^s}{R^{s+n+1}} (-1)^{n+m} P_s{}^m(\cos\theta), \tag{7b}$$

for $0 < r_1,\ r < R$.

In order to satisfy the boundary conditions at the surface of sphere O, eqn (5) should be expressed in terms of the (r, θ, α) coordinates. By using $z_1 = z + R$ and (7), we obtain

$$2\mu_0\mathbf{u} = 2\mu_0\mathbf{u}^\infty + \text{grad}(\zeta - R\,\psi_1) + \text{grad}\,[z(\psi + \psi_1)] - 4(1-\nu_0)(\psi - \psi_1)\mathbf{e}_z \tag{8}$$

and

$$\zeta - R\psi_1 = \sum_{n=0}^{\infty}\left\{A_n \frac{a^{n+3}}{r^{n+1}} P_n(\cos\theta) + (A_n{}^1 a - RC_n{}^1)a^{n+2} \sum_{s=0}^{\infty} g_{ns}(-1)^n r^s P_s(\cos\theta)\right\}$$

$$\psi + \psi_1 = \sum_{n=0}^{\infty}\left\{C_n \frac{a^{n+2}}{r^{n+1}} P_n(\cos\theta) + C_n{}^1 a^{n+2} \sum_{s=0}^{\infty} g_{ns}(-1)^n r^s P_s(\cos\theta)\right\}$$

where

$$g_{ns} = \frac{(s+n)!}{s!\,n!} R^{-(s+n+1)}.$$

The displacement and stress fields corresponding to (8) in spherical coordinates become then

$$u_r = u_r^\infty + \frac{1}{2\mu_0}\sum_{n=0}^{\infty}\left\{-A_n\frac{(n+1)a^{n+3}}{r^{n+2}}P_n - C_n\frac{a^{n+2}}{r^{n+1}}\frac{n+4-4\nu_0}{2n+1}\right.$$
$$\times[(n+1)P_{n+1}+nP_{n-1}] + (A_n{}^1 a - RC_n{}^1)a^{n+2}\sum_{s=0}^{\infty}g_{ns}(-1)^n r^{s-1}sP_s + C_n{}^1 a^{n+2}$$
$$\left.\times\sum_{s=0}^{\infty}g_{ns}(-1)^n r^s\frac{s-3+4\nu_0}{2s+1}[(s+1)P_{s+1}+sP_{s-1}]\right\}$$

$$u_\theta = u_\theta^\infty - \frac{\sin\theta}{2\mu_0}\sum_{n=0}^{\infty}\left\{A_n\frac{a^{n+3}}{r^{n+2}}P_n' + C_n\frac{a^{n+2}}{r^{n+1}}\frac{1}{2n+1}\right.$$
$$\times[(n-3+4\nu_0)P_{n+1}'+(n+4-4\nu_0)P_{n-1}'] + (A_n{}^1 a - RC_n{}^1)a^{n+2}\sum_{s=0}^{\infty}g_{ns}(-1)^n r^{s-1}P_s' + C_n{}^1 a^{n+2}$$
$$\left.\times\sum_{s=0}^{\infty}g_{ns}(-1)^n r^s\frac{1}{2s+1}[(s-3+4\nu_0)P_{s+1}'+(s+4-4\nu_0)P_{s-1}']\right\} \tag{9}$$

$$u_\alpha = 0$$

and

$$\sigma_{rr} = \sigma_{rr}^\infty + \sum_{n=0}^{\infty}\left\{A_n(n+1)(n+2)\frac{a^{n+3}}{r^{n+3}}P_n + C_n\frac{a^{n+2}}{r^{n+2}}\frac{n+1}{2n+1}\right.$$
$$\times[(n+4-4\nu_0)nP_{n-1}+(n^2+5n+4-2\nu_0)P_{n+1}] + (A_n{}^1 a - RC_n{}^1)a^{n+2}$$
$$\times\sum_{s=0}^{\infty}g_{ns}(-1)^n s(s-1)r^{s-2}P_s + C_n{}^1 a^{n+2}\sum_{s=0}^{\infty}g_{ns}(-1)^n r^{s-1}\frac{s}{2s+1}$$
$$\left.\times[(s^2-3s-2\nu_0)P_{s-1}+(s+1)(s-3+4\nu_0)P_{s+1}]\right\}$$

$$\sigma_{r\theta} = \sigma_{r\theta}^\infty + \sin\theta\sum_{n=0}^{\infty}\left\{A_n(n+2)\frac{a^{n+3}}{r^{n+3}}P_n' + C_n\frac{a^{n+2}}{r^{n+2}}\frac{1}{2n+1}\right.$$
$$\times[(n^2+2n-1+2\nu_0)P_{n+1}'+(n+1)(n+4-4\nu_0)P_{n-1}'] - (A_n{}^1 a - RC_n{}^1)a^{n+2}$$
$$\left.\times\sum_{s=0}^{\infty}g_{ns}r^{s-2}(-1)^n(s-1)P_s' - C_n{}^1 a^{n+2}\sum_{s=0}^{\infty}g_{ns}\frac{r^{s-1}}{2s+1}[(s-3+4\nu_0)sP_{s+1}'+(s^2-2+2\nu_0)P_{s-1}']\right\}$$

$$\sigma_{r\alpha} = 0. \tag{10}$$

Similarily, we obtain the expressions for the displacement and for the stress fields inside sphere O,

$$u_r^{(p)} = \frac{1}{2\mu_p}\sum_{n=0}^{\infty}\left\{D_n\frac{r^n}{a^{n-1}}\frac{n-3+4\nu_p}{2n+1}[(n+1)P_{n+1}+nP_{n-1}] + B_n\frac{nr^{n-1}}{a^{n-2}}P_n\right\}$$

$$u_\theta^{(p)} = \frac{-\sin\theta}{2\mu_p}\sum_{n=0}^{\infty}\left\{D_n\frac{r^n}{a^{n-1}}\frac{1}{2n+1}[(n-3+4\nu_p)P_{n+1}'+(n+4-4\nu_p)P_{n-1}'] + B_n\frac{r^{n-1}}{a^{n-2}}P_n'\right\}$$

$$u_\alpha^{(p)} = 0 \tag{11}$$

and

$$\sigma_{rr}^{(p)} = \sum_{n=0}^{\infty}\left\{D_n\frac{r^{n-1}}{a^{n-1}}\left[\frac{n(n+1)(n-3+4\nu_p)}{2n+1}P_{n+1} + \frac{n(n^2-3n-2\nu_p)}{2n+1}P_{n-1}\right] + B_n\frac{n(n-1)r^{n-2}}{a^{n-2}}P_n\right\}$$

$$\sigma_{r\theta}^{(p)} = -\sin\theta \sum_{n=0}^{\infty} \left\{ D_n \frac{r^{n-1}}{a^{n-1}} \frac{1}{2n+1} [(n-3+4\nu_p)\, nP'_{n+1} + (n^2-2+2\nu_p)\, P'_{n-1}] + B_n \frac{r^{n-2}}{a^{n-2}} (n-1) P'_n \right\} \tag{12}$$

$$\sigma_{r\alpha}^{(p)} = 0.$$

The expressions for the remaining stresses, which are not shown here, have been derived by Chen[7]. From the definition of the stresslet S_{ij} and the exterior solution, we can then easily calculate that, for a sphere of radius a,

$$S_{ij} = \frac{4\pi a^3(1-\nu_0)}{1-2\nu_0} [A_0\delta_{ij} + C_1(\delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2} + (3-4\nu_0)\delta_{i3}\delta_{j3})] \tag{13a}$$

and

$$S_{ii} = \frac{4\pi a^3(1-\nu_0)}{1-2\nu_0} (3A_0 + (5-4\nu_0)C_1). \tag{13b}$$

Since the loading, given by (4), is symmetric both with respect to the axis of symmetry of the two spheres and, for equal sized spheres, to the plane of symmetry perpendicular to this axis, we have that $A_n = (-1)^n A_n{}^1$ and $C_n = (-1)^{n+1} C_n{}^1$. The two-sphere problem then simplifies to that of finding the unknown coefficients A_n and C_n by satisfying the boundary conditions at the surface of sphere O.

Substituting the exterior and the interior solutions (9)–(12) in the boundary conditions involving the continuity of displacement and traction, and using the orthogonality of the Legendre polynomials, we can obtain a set of relations for the coefficients A_n, B_n, C_n and D_n [7]. However, since the calculation of the effective bulk modulus κ^* involves only S_{ii} [6] which depends only on A_0 and C_1, we eliminate the interior coefficients B_n and D_n and obtain

$$A_0 + \frac{5-4\nu_0}{3} C_1 = 2\mu_0\gamma_1 + \frac{2}{3}(1-2\nu_0)\gamma_1 \sum_{s=0}^{\infty} C_s(s+1)\rho^{s+2}$$

and for $n \geq 0$

$$A_n = 2\mu_0\gamma_1\delta_{n0} + \sum_{s=0}^{\infty} a^{s+n}\{(A_s a + RC_s)(M_1 g_{sn} + M_2 a^2 g_{s(n+2)}) - C_s(M_3 g_{s(n-1)} + M_4 a^2 g_{s(n+1)} + M_5 a^4 g_{s(n+3)})\} \tag{14a}$$

$$C_n = -\sum_{s=0}^{\infty} a^{s+n+1}\{M_6(A_s a + RC_s) g_{s(n+1)} - C_s(M_7 g_{sn} + M_8 a^2 g_{s(n+2)})\}, \tag{14b}$$

where $\rho \equiv (a/R)$, $\gamma(3\kappa_p - 3\kappa_0/3\kappa_p + 3\mu_0)$, $g_{ns} = ((s+n)!/s!n!)\, R^{-(s+n+1)}$, and all of the M_i's are given the Appendix. The above linear equations can then be solved by expanding A_n and C_n into power series in ρ, and obtaining a recurrence formula for the appropriate coefficients. For reference, an example of this method is also illustrated in the Appendix. The first few terms of the coefficients are

$$A_0^* = 2 + \frac{5(1-\beta)(5-4\nu_0)}{2\beta(4-5\nu_0) + (7-5\nu_0)} \rho^3 - \left[\frac{50(1-\beta)^2(5-4\nu_0)(2-\nu_0)}{[2\beta(4-5\nu_0)+(7-5\nu_0)]^2} + \frac{20(1-\nu_0)(1-2\nu_0)}{2\beta(4-5\nu_0)+(7-5\nu_0)} \gamma_1 \right] \rho^6 + 0(\rho^8)$$

$$A_1^* = \frac{28(3-2\nu_0)(1-\beta)}{2\beta(11-14\nu_0) + (13-7\nu_0)} \rho^4 + 0(\rho^6)$$

$$A_n^* = \frac{n(n-1)(2n-1)(1-\beta)}{[\beta(n-1)(3n+2-4n\nu_0-2\nu_0) + (n^2+n+1-2n\nu_0-\nu_0)]} \rho^{n+1} + 0(\rho^{n+3})$$

and

$$C_0^* = 0$$

$$C_1^* = \frac{15(\beta-1)}{2\beta(4-5\nu_0)+(7-5\nu_0)}\rho^3 + \frac{150(1-\beta)^2(2-\nu_0)}{[2\beta(4-5\nu_0)+(7-5\nu_0)]^2}\rho^6 + 0(\rho^8)$$

$$C_n^* = -\frac{n(2n+1)(2n-1)(1-\beta)}{\beta n(3n+5-4n\nu_0-6\nu_0)+(n^2+3n+3-2n\nu_0-3\nu_0)}\rho^{n+2} + 0(\rho^{n+5})$$

where

$$A_n^* = A_n/(\mu_0\gamma_1), \quad C_n^* = C_n/(\mu_0\gamma_1) \text{ and } \beta = \mu_p/\mu_0.$$

4. AXISYMMETRIC SOLUTION FOR $\epsilon_{ij}^\infty = \delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2} - 2\delta_{i3}\delta_{j3}$

The displacement and the stress fields corresponding to the above strain at infinity are (with x_1^* taken from the midpoint of the two centers of the spheres)

$$u_i^\infty = x_1^*\delta_{i1} + x_2^*\delta_{i2} - 2x_3^*\delta_{i3}$$

and

$$\sigma_{ij}^\infty = 2\mu_0(\delta_{i1}\delta_{j1} + \delta_{i1}\delta_{j2} - 2\delta_{i3}\delta_{j3}).$$

Since the system is also axisymmetric, the method of solution is exactly the same as that given in Section 3, the only difference being in the field applied at infinity. The solutions for the unknown coefficients A_n and C_n are therefore similar to eqns (14a, b) except that the terms before the summation signs are $5\mu_0(5-4\nu_0)\gamma_2\delta_{n0} + 6\mu_0\gamma_2\delta_{n2}$ in the equation for A_n and $-15\mu_0\gamma_2\delta_{n1}$ in the equation for C_n, where $\gamma_2 = (\beta-1)/[2\beta(4-5\nu_0)+(7-5\nu_0)]$. The first terms of the coefficients become then:

$$C_0^+ = 0$$

$$C_1^+ = -15 + \frac{150(1-\beta)(2-\nu_0)}{2\beta(4-5\nu_0)+(7-5\nu_0)}\rho^3 - \frac{540(1-\beta)}{2\beta(4-5\nu_0)+(7-5\nu_0)}\rho^5$$

$$+ [900\gamma_2^2(2-\nu_0)(1-\nu_0) - 150(1-2\nu_0)\gamma_1\gamma_2]\rho^6 + O(\rho^8)$$

$$C_n^+ = \frac{5(1-\beta)n(2n+3)(n+2)(3n+5-4\nu_0)}{2[\beta n(3n-4n\nu_0+5-6\nu_0)+(n^2+3n+3-2n\nu_0-3\nu_0)]}\rho^{n+2} + O(\rho^{n+4})$$

and

$$A_0^+ = 5(5-4\nu_0) + \left[-\frac{50(5-4\nu_0)(2-\nu_0)(1-\beta)}{2\beta(4-5\nu_0)+(7-5\nu_0)} - 20\gamma_1(1-2\nu_0)\right]\rho^3 + O(\rho^5)$$

$$A_1^+ = O(\rho^4)$$

$$A_2^+ = 6 + \frac{60(\beta-1)(2-\nu_0)}{2\beta(4-5\nu_0)+(7-5\nu_0)}\rho^3 + O(\rho^5)$$

$$A_n^+ = -\frac{5(1-\beta)(n-1)n(n+1)(3n+2-4\nu_0)}{2[\beta(n-1)(3n-4n\nu_0+2-2\nu_0)+(n^2+n+1-2n\nu_0-\nu_0)]}\rho^{n+1} + O(\rho^{n+3})$$

where

$$A_n^+ = A_n/(\mu_0\gamma_2) \text{ and } C_n = C_n/(\mu_0\gamma_2).$$

5. ASYMMETRIC SOLUTION FOR $\epsilon_{ij}^\infty = \delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1}$

The corresponding displacement and stress at infinity are

$$u_i^\infty = x_2^*\delta_{i1} + x_1^*\delta_{i2}$$

and

$$\sigma_{ij}^{\infty} = 2\mu_0(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1}).$$

Transforming to the center of sphere O in spherical coordinates yields the displacements

$$u_r^{\infty} = \frac{r}{3} P_2^2(\cos\theta)\sin 2\alpha$$

$$u_\theta^{\infty} = -\frac{r\sin\theta}{6} P_2^{2\prime}\sin 2\alpha \tag{15a}$$

$$u_\alpha^{\infty} = \frac{r}{3\sin\theta} P_2^2 \cos 2\alpha$$

and the three stresses in the radial direction

$$\sigma_{rr}^{\infty} = \frac{2\mu_o}{3} P_2^2 \sin 2\alpha \quad \sigma_{r\theta}^{\infty} = -\frac{\mu_0\sin\theta}{3} P_2^{2\prime}\sin 2\alpha$$

$$\sigma_{r\alpha}^{\infty} = \frac{2\mu_0}{3\sin\theta} P_2^2\cos 2\alpha \tag{15b}$$

where the P_n^m's are the associated Legendre polynomials and $P_n^{m\prime}(x) = \mathrm{d}P_n^m(x)/\mathrm{d}x$. To satisfy the conditions at infinity, we take into account that the displacements and stresses are all proportional to $\cos 2\alpha$ or $\sin 2\alpha$, and choose the four stress functions as follows:

$$\psi = \psi^*\sin 2\alpha$$

$$\tau_1 = \tau_1^*\sin\alpha$$

$$\tau_2 = \tau_1^*\cos\alpha \tag{16}$$

$$\tau_3 = \tau_3^*\sin 2\alpha$$

where ψ^*, τ_1^* and τ_3^* depend on r and θ only. In eqn (16), the four stress functions are combined to yield three independent functions. In view of (16), the solution of the displacement field outside the two sphere can then be represented as

$$2\mu_0\mathbf{u} = 2\mu_0\mathbf{u}^{\infty} + \mathrm{grad}(\psi^*\sin 2\alpha) + \mathrm{grad}(x\tau\sin\alpha + y\tau\cos\alpha + z\xi\sin 2\alpha$$

$$+ z_1\xi_1\sin 2\alpha) - 4(1-\nu_0)[\tau\sin\alpha, \tau\cos\alpha, (\xi + \xi_1)\sin 2\alpha], \tag{17}$$

where

$$\psi^* = \sum_{n=2}^{\infty}\left\{A_n \frac{a^{n+3}}{r^{n+1}} P_n^2(\cos\theta) + A_n^1 \frac{a^{n+3}}{r_1^{n+1}} P_n^2(\cos\theta_1)\right\}$$

$$\tau = \sum_{n=1}^{\infty}\left\{B_n \frac{a^{n+2}}{r^{n+1}} P_n^1(\cos\theta) + B_n^1 \frac{a^{n+2}}{r_1^{n+1}} P_n^1(\cos\theta_1)\right\}$$

$$\xi = \sum_{n=2}^{\infty} C_n \frac{a^{n+2}}{r^{n+1}} P_n^2(\cos\theta), \quad \xi_1 = \sum_{n=2}^{\infty} C_n^1 \frac{a^{n+2}}{r^{n+1}} P_n^2(\cos\theta_1).$$

Since we have to satisfy the boundary conditions on the surfaces of both particles, we shall first express (17) in terms of the coordinates (r, θ, α) with respect to spheres O and O_1, using $z = z_1 + R$ and the relations between the harmonic functions referred to each of the two spheres (7). Because the spheres are identical, the system is symmetric with respect to the plane passing through the midpoint of the centers of spheres and perpendicular to the common axis of the system. Therefore, we obtain

$$A_n = (-1)^n A_n^1, \quad B_n = (-1)^{n+1} B_n^1, \quad C_n = (-1)^{n+1} C_n^1.$$

Thus, the problem is simplified and only the boundary conditions on one of the spheres have to be satisfied. We shall now proceed with the solutions for the limiting cases of either rigid particles or cavities.

In the case of rigid inclusions, the displacements on the surfaces of the spheres vanish, i.e. $u_i = 0$, because of the symmetry of the problem. Upon satisfying this condition on the surface of sphere O, we can then obtain[7] a set of equations relating A_n, B_n and C_n which were solved to give the first few terms of the series in ρ

$$A_2{}^I = -1 + \frac{5(1-2\nu_0)}{2(4-5\nu_0)}\rho^3 - \left[\frac{1-5\nu_0}{4-5\nu_0} + \frac{5(7+4\nu_0)}{7-9\nu_0}\right]\rho^5 + O(\rho^6)$$

$$A_n{}^I = \frac{-5(n-4+4\nu_0)}{2(3n+2-4n\nu_0-2\nu_0)}\rho^{n+1} + O(\rho^{n+3})$$

$$B_1{}^I = 5 - \frac{25(1-2\nu_0)}{2(4-5\nu_0)}\rho^3 - \frac{15}{4-5\nu_0}\rho^5 + \frac{125(1-2\nu_0)^2}{4(4-5\nu_0)^2}\rho^6 + O(\rho^8)$$

$$B_2{}^I = -\frac{10(3-7\nu_0)}{11-14\nu_0}\rho^4 - \frac{60}{11-14\nu_0}\rho^6 + O(\rho^7)$$

$$B_n{}^I = -\frac{5(n+4-4n\nu_0-6\nu_0)}{3n+5-4n\nu_0-6\nu_0}\rho^{n+2} + O(\rho^{n+4})$$

and

$$C_2{}^I = \frac{25}{2(11-14\nu_0)}\rho^4 - \frac{30}{11-14\nu_0}\rho^6 + O(\rho^7)$$

$$C_n{}^I = \frac{5(2n+1)}{2(3n+5-4n\nu_0-6\nu_0)}\rho^{n+2} + O(\rho^{n+4})$$

where the superscript I denotes multiplication of the corresponding quantity by $2(4-5\nu_0)/\mu_0$.

When the inclusions are cavities, the proper boundary condition becomes $\sigma_{ij}n_j = 0$, where n_j is the unit normal vector of the surfaces. In terms of the spherical coordinates of Fig. 1 we have

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{r\alpha} = 0 \qquad \text{at } r = a.$$

After satisfying the boundary conditions, we again obtain another set of equations for the coefficients A_n, B_n and C_n which were solved[7] to yield the first few terms of the series in ρ

$$A_2^{II} = 1 + \frac{5(1-2\nu_0)}{7-5\nu_0}\rho^3 + O(\rho^5)$$

$$A_n^{II} = -\frac{5(n-1)(n-4+4\nu_0)}{2(n^2+n-2n\nu_0+1-\nu_0)}\rho^{n+1} + O(\rho^{n+3})$$

$$B_1^{II} = -5 - \frac{25(1-2\nu_0)}{7-5\nu_0}\rho^3 - \frac{30}{7-5\nu_0}\rho^5 - \frac{125(1-2\nu_0)^2}{(7-5\nu_0)^2}\rho^6 - \left[\frac{300(1-2\nu_0)}{7-5\nu_0} + \frac{75(9-44\nu_0+77\nu_0^2)}{2(13-7\nu_0)(7-5\nu_0)}\right]\rho^8 + O(\rho^9)$$

$$B_2^{II} = -\frac{5(23-77\nu_0)}{4(13-7\nu_0)}\rho^4 - \frac{120}{13-7\nu_0}\rho^6 + O(\rho^7)$$

$$B_n^{II} = \frac{5(n^3-n^2+10n^2\nu_0-12n+17n\nu_0-3+3\nu_0)}{(n+2)(n^2+3n-2n\nu_0+3-3\nu_0)}\rho^{n+2} + O(\rho^{n+4})$$

and

$$C_2^{II} = -\frac{25(1-7\nu_0)}{4(13-7\nu_0)}\rho^4 - \frac{60}{13-7\nu_0}\rho^6 + O(\rho^7)$$

$$C_n^{II} = \frac{5(2n+1)(n^2+4n\nu_0-6+6\nu_0)}{2(n+2)(n^2+3n-2n\nu_0+3-3\nu_0)}\rho^{n+2} + O(\rho^{n+4})$$

where the superscript II denotes multiplication of the corresponding quantity by $(7-5\nu_0)/\mu_0$. For this applied strain, the only non-zero component of the stresslet on the reference sphere of radius a can also be evaluated to be

$$S_{21} = S_{12} = 8\pi a^3(1-\nu_0)B_1.$$

6. ASYMMETRIC SOLUTION FOR $\epsilon_{ij}^{\infty} = \delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2}$

For this applied strain, the displacement and stress at infinity are

$$u_i^{\infty} = x_2^*\delta_{i3} + x_3^*\delta_{i2}$$

and

$$\sigma_{ij}^{\infty} = 2\mu_0(\delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2}).$$

In the new coordinates with the origin being at the center of sphere O, the above expressions become

$$u_r^{\infty} = \left(\frac{2r}{3}P_2{}^1 - \frac{R}{2}P_1{}^1\right)\sin\alpha, \quad u_\theta^{\infty} = \left(-\frac{r}{3}P_2{}^{1\prime} + \frac{R}{2}P_1{}^{1\prime}\right)\sin\theta\,\sin\alpha,$$

$$u_\alpha^{\infty} = \left(\frac{r}{3}P_2{}^1 - \frac{R}{2}P_1{}^1\right)\frac{\cos\alpha}{\sin\theta} \tag{18a}$$

and

$$\sigma_{rr}^{\infty} = \frac{4\mu_0}{3}P_2{}^1\sin\alpha, \quad \sigma_{r\theta}^{\infty} = -\frac{2\mu_0}{3}P_2{}^{1\prime}\sin\theta\,\sin\alpha,$$

$$\sigma_{r\alpha}^{\infty} = \frac{2\mu_0}{3\sin\theta}P_2{}^1\cos\alpha, \text{etc.} \tag{18b}$$

Since the displacements and stresses are all proportional to either $\sin\alpha$ or $\cos\alpha$, the four stress functions are chosen as

$$\psi = \psi^*(r,\theta)\sin\alpha, \quad \tau_1 = 0,$$

$$\tau_2 = \tau_2^*(r,\theta), \qquad \tau_3 = \tau_3^*(r,\theta)\sin\alpha.$$

Using the "multipole expansion" method, the displacement field outside the two spheres can then be expressed as follows:

$$2\mu_0\mathbf{u} = 2\mu_0\mathbf{u}^{\infty} + \text{grad}(\psi^*\sin\alpha) + \text{grad}(y\tau + z\xi\sin\alpha + z_1\xi_1\sin\alpha) - 4(1-\nu_0)\,[0,\tau,(\xi+\xi_1)\sin\alpha], \tag{19}$$

where

$$\psi^* = \sum_{n=1}^{\infty}\left\{A_n\frac{a^{n+3}}{r^{n+1}}P_n{}^1(\cos\theta) + A_n{}^1\frac{a^{n+3}}{r_1^{n+1}}P_n{}^1(\cos\theta_1)\right\}$$

$$\tau = \sum_{n=0}^{\infty}\left\{B_n\frac{a^{n+2}}{r^{n+1}}P_n(\cos\theta) + B_n{}^1\frac{a^{n+2}}{r_1^{n+1}}P_n(\cos\theta_1)\right\}$$

$$\xi = \sum_{n=1}^{\infty}C_n\frac{a^{n+2}}{r^{n+1}}P_n{}^1(\cos\theta), \quad \xi_1 = \sum_{n=1}^{\infty}C_n{}^1\frac{a^{n+2}}{r_1^{n+1}}P_n{}^1(\cos\theta_1).$$

We shall again follow the procedures given in the previous sections by transforming the origin to the center of the sphere O and satisfying the boundary conditions. However, unlike the previous cases, there is no plane of symmetry in this system. Nevertheless, we find that the solution of this case should be symmetric with respect to the midpoint of the centers of spheres. Thus, we have

$$u_r(r,\theta,\alpha) = u_r'(r,\pi-\theta,\pi+\alpha)$$

$$u_\theta(r,\theta,\alpha) = u_\theta'(r,\pi-\theta,\pi+\alpha)$$

$$u_\alpha(r,\theta,\alpha) = u_\alpha'(r,\pi-\theta,\pi+\alpha)$$

where the primes denote the quantities referred to the origin at O_1. Therefore, we have $A_n = (-1)^n A_n^1$, $B_n = (-1)^{n+1} B_n^1$ and $C_n = (-1)^{n+1} C_n^1$ which simplifies the two-sphere problem.

In order to solve for the unknown coefficients A_n, B_n and C_n, we have to satisfy the boundary conditions on the surface of sphere O. For the case of rigid inclusions, the displacement has to be continuous and the net force and torque must vanish. Since on the particle O,

$$\text{total force} = \int \sigma_{ij} n_j \mathrm{d}S = 8\pi a^2(1-\nu_0) B_0 \delta_{i2}$$

$$\text{and total torque} = \int \epsilon_{ijk} n_j \sigma_{km} n_m \mathrm{d}S = 8\pi a^2 (C_1 - B_1)\delta_{i1},$$

we conclude that $B_0 = 0$ and $C_1 = B_1$. This condition is necessary for obtaining a unique solution for rigid particles, but not for cavities. The rigid particle displacement on sphere O under the applied strain $\epsilon_{ij}^\infty = \delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2}$ can be expressed in the form

$$u_i^{(p)} = V^{(p)}\delta_{i2} + a\Omega^{(p)}\,\epsilon_{i1k} n_k,$$

where the function $V^{(p)}$ and $\Omega^{(p)}$ depend on $\rho \equiv a/R$ and the elastic moduli of the matrix.

After satisfying the conditions on the surface of sphere O, a set of relations for A_n, B_n and C_n was obtained which were solved in[7] using the method described previously. The first few terms of the coefficients are

$$B_1^I = 5 - \frac{25(1+\nu_0)}{2(4-5\nu_0)}\rho^3 + \frac{60}{4-5\nu_0}\rho^5 + \frac{125(1+\nu_0)^2}{4(4-5\nu_0)^2}\rho^6 + \left[\frac{150(18-23\nu_0+14\nu_0^2)}{(4-5\nu_0)(11-14\nu_0)} - \frac{225(4+\nu_0)}{2(4-5\nu_0)^2}\right]\rho^8 + O(\rho^9)$$

$$B_n^I = \frac{5[(n+1)(4n+9-8n\nu_0-12\nu_0)-(n+3)(2n+1)]}{2(3n+5-4n\nu_0-6\nu_0)}\rho^{n+2} + \frac{2(n+3)(n+4)(2n+1)}{3n+5-4n\nu_0-6\nu_0}\rho^{n+4} + O(\rho^{n+5})$$

$$C_1^I = B_1^I$$

$$C_n^I = \frac{5[2n^2-5n+8n\nu_0-10+12\nu_0-(3n+4)(2n+1)]}{2(3n+5-4n\nu_0-6\nu_0)}\rho^{n+2} + \frac{2(n+3)(n+4)(2n+1)}{3n+5-4n\nu_0-6\nu_0}\rho^{n+4} + O(\rho^{n+5})$$

and

$$A_1^I = \left[10 + \frac{40(3-2\nu_0)}{11-14\nu_0}\right]\rho^4 - \frac{120(3-2\nu_0)}{11-14\nu_0}\rho^6 + O(\rho^7)$$

$$A_2^I = -2 + \frac{5(1+\nu_0)}{4-5\nu_0}\rho^3 + \left[\frac{25(7-4\nu_0)^2}{2(7-9\nu_0)^2} + \frac{8(1-5\nu_0)}{4-5\nu_0}\right]\rho^5 - \frac{25(1+\nu_0)^2}{2(4-5\nu_0)^2}\rho^6 + O(\rho^7)$$

$$A_n^I = \frac{5(n^2-2+2\nu_0)}{3n+2-4n\nu_0-2\nu_0}\rho^{n+1} + \left[\frac{5(n+3)(n+5-4\nu_0)}{3n+8-4n\nu_0-10\nu_0} - \frac{2(n+2)(n^2+2n-10+10\nu_0)}{3n+2-4n\nu_0-2\nu_0}\right]\rho^{n+3} + O(\rho^{n+4})$$

where, as before, the superscript denotes multiplication of the corresponding quantity by $2(4-5\nu_0)/\mu_0$. The rigid particle displacement and rotation can also be calculated from the relations $B_0 = 0$ and $C_1 = B_1$ to be

$$\frac{2\mu_0 V^{(p)}}{a} = -\frac{R}{a}\mu_0 - \sum_{s=1}^{\infty}\left\{A_s\rho^{s+2} - B_s\rho^{s+1}\left[(3-4\nu_0) + \frac{(s+2)(2+1)}{6}\rho^2\right] + C_s\rho^{s+1}[1-(s+2)\rho^2]\right\}$$

and

$$\mu_0\Omega^{(p)} = -(1-\nu_0)\sum_{s=1}^{\infty}\{(s+1)B_s + C_s\}\rho^{s+2}.$$

Similarly, in the case of cavities, the first few coefficients become

$$B_1^{II} = -5 - \frac{25(1+\nu_0)}{7-5\nu_0}\rho^3 + \frac{120}{7-5\nu_0}\rho^5 - \frac{125(1+\nu_0)^2}{(7-5\nu_0)^2}\rho^6 + \left[\frac{15(1+\nu_0)}{7-5\nu_0} + \frac{1200(1+\nu_0)}{(7-5\nu_0)^2} - \frac{30(139+38\nu_0+14\nu_0^2)}{(7-5\nu_0)(13-7\nu_0)}\right]\rho^8 + O(\rho^9)$$

$$B_n^{II} = -\frac{5n(n^2+4n+2n\nu_0+3\nu_0)}{n^2+3n-2n\nu_0+3-3\nu_0}\rho^{n+2} + \frac{2n(n+3)(n+4)(2n+1)}{n^2+3n-2n\nu_0+3-3\nu_0}\rho^{n+4} + O(\rho^{n+5})$$

$$C_1^{II} = B_1^{II}$$

$$C_n^{II} = -\frac{5(2n^3+6n^2+2n\nu_0-3+3\nu_0)}{n^2+3n-2n\nu_0+3-3\nu_0}\rho^{n+2} + \frac{2n(n+3)(n+4)(2n+1)}{n^2+3n-2n\nu_0+3-3\nu_0}\rho^{n+4} + O(\rho^{n+5})$$

and

$$A_1^{II} = \left[\frac{4(3-2\nu_0)(12+7\nu_0)}{13-7\nu_0} + (7-8\nu_0)\right]\rho^4 + O(\rho^6)$$

$$A_2^{II} = 2 + \frac{10(1+\nu_0)}{7-5\nu_0}\rho^3 + O(\rho^5)$$

$$A_n^{II} = \frac{5(n-1)(n^2-2+2\nu_0)}{n^2+n-2n\nu_0+1-\nu_0}\rho^{n+1} + O(\rho^{n+3})$$

where, as before, A_n^{II}, B_n^{II} and C_n^{II} are defined as A_n, B_n and C_n multiplied by the quantity $(7-5\nu_0)/\mu_0$. The stresslet for this applied strain can be calculated to give $S_{ij} = 8\pi a^3(1-\nu_0)B_1(\delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2})$.

7. NUMERICAL RESULTS

With Poisson's ratio of the matrix being chosen equal to 0.25, numerical calculations were performed for an infinite region containing either two rigid particles or two cavities which yielded the stresses along the centerline of the spheres and the displacement on the rigid particles.

Generally, the stresses at any given point outside the spheres will be a function only of the applied strain at infinity ϵ_{ij}^{∞}, the position vector of the point x_i (the origin being at the center of sphere O), and the orientation vector y_i of the two-sphere system. It is easy to show, moreover, from the linearity of the problem, that when $x_i = r\,\delta_{i3}$ and $y_i = R\delta_{i3}$, the stress can be expressed as

$$\sigma_{ij} = \lambda_1\epsilon_{kk}\delta_{ij} + \lambda_2\epsilon_{33}\delta_{ij} + \lambda_3\epsilon_{kk}\delta_{i3}\delta_{j3} + \lambda_4(\epsilon_{ij} - \tfrac{1}{3}\epsilon_{kk}\delta_{ij}) + \lambda_5(\delta_{i3}\epsilon_{j3} + \delta_{j3}\epsilon_{i3} - \tfrac{2}{3}\epsilon_{33}\delta_{ij}) + \lambda_6\epsilon_{33}(\delta_{i3}\delta_{j3} - \tfrac{1}{3}\delta_{ij})$$

where the λ_i's (i = 1–6) depend only on R/a and r, and on the elastic parameters of the matrix and the inclusions. To evaluate these coefficients, the same four systems (1a to 1d) have to be solved. The results of the numerical calculations for the stresses are shown in Fig. 2–7 for the section between the spheres, which is the region where the interaction due to the other sphere is most significant. The stresses not shown in the figures are all equal to zero.

Similarily, the displacement and the rotation on the rigid sphere O can be expressed as

$$\frac{u_i}{a} = \lambda_7\epsilon_{i3} + \lambda_8\epsilon_{33}\delta_{i3} + \lambda_9\epsilon_{kk}\delta_{i3}$$

and

$$\Omega_i = \lambda_{10}\epsilon_{ij3}\epsilon_{j3},$$

(ϵ_{ijk} being the well-known permutation symbol) where the coefficients $\lambda_7 - \lambda_{10}$, which are now functions only of R/a, can be obtained from the solutions of systems (1a), (1b) and (1d) and are shown in Fig. 8.

As expected, the series solutions developed in the present work converge very rapidly when the spheres are far apart. Thus, generally for $R/a \geq 3.0$, very accurate results were obtained using $n = 30$ for the calculations of the coefficients A_n, B_n, C_n and for the stresses. However, as the spheres approach each other, the accuracy using the present technique decreases rapidly and little improvement could be attained by increasing the number of terms used to $n = 70$ even though the series no doubt remains convergent for all $(R/a) > 2$. Some of the computed stresses on the surface of the cavities are shown in Table 1 for comparison with the exact solution which results readily from the conditions of zero traction. Since the entries in the second, third and fourth columns of Table 1 should read, respectively, −5, 4 and −2 for all $(R/a) \geq 2$, the loss

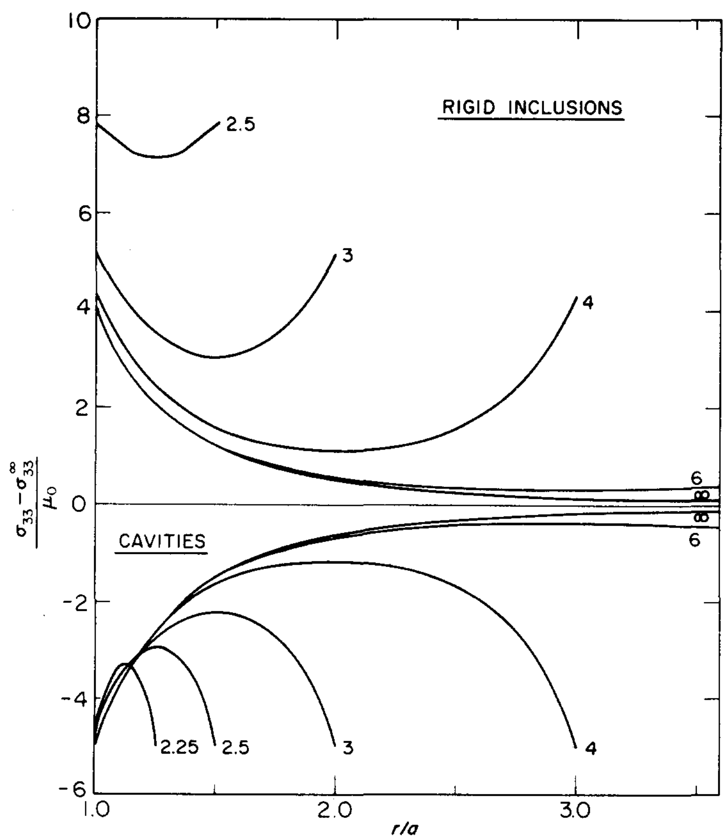

Fig. 2. Stresses along the centerline of the spheres for the applied strain $\epsilon_{ij}^{\infty} = \delta_{ij}$. The number beside each curve denotes the value of R/a.

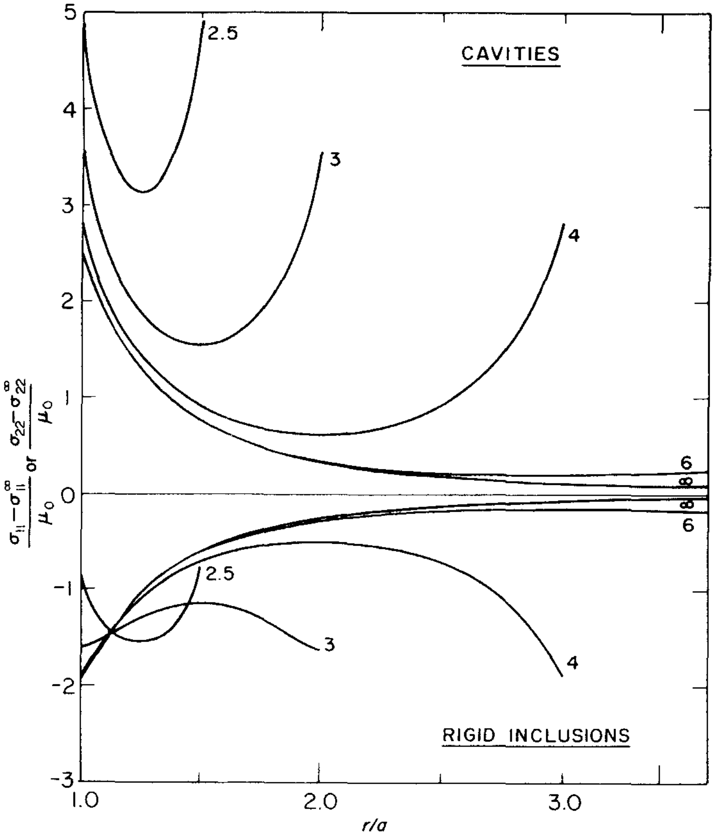

Fig. 3. Stresses along the centerline of the spheres for the applied strain $\epsilon_{ij}^{\infty} = \delta_{ij}$.

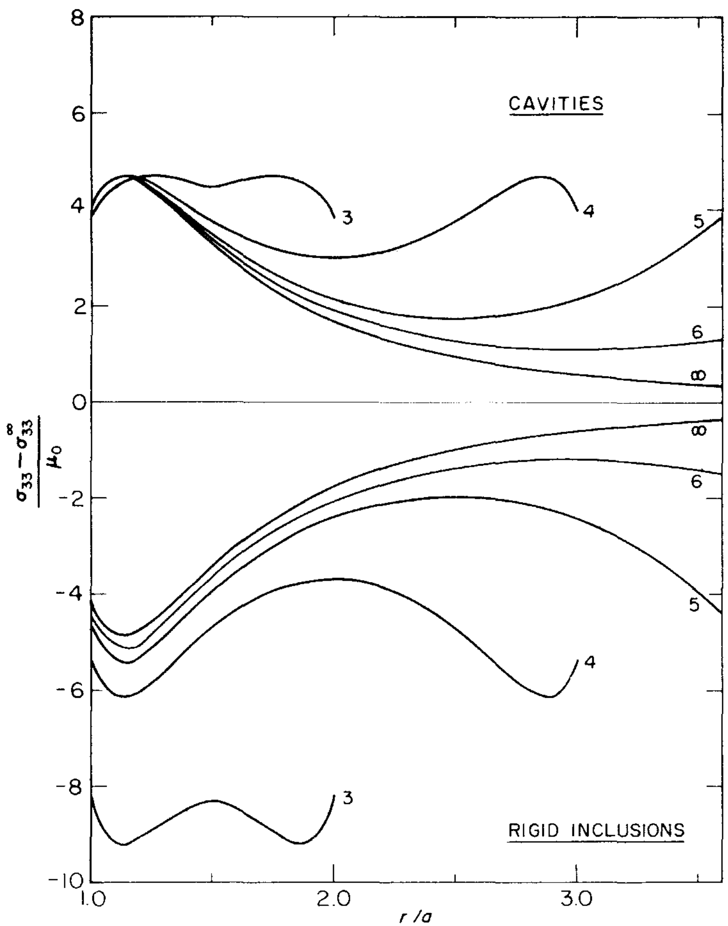

Fig. 4. Stresses along the centerline of the spheres for the applied strain $\epsilon_{ij}^{\infty} = \delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2} - 2\delta_{i3}\delta_{j3}$.

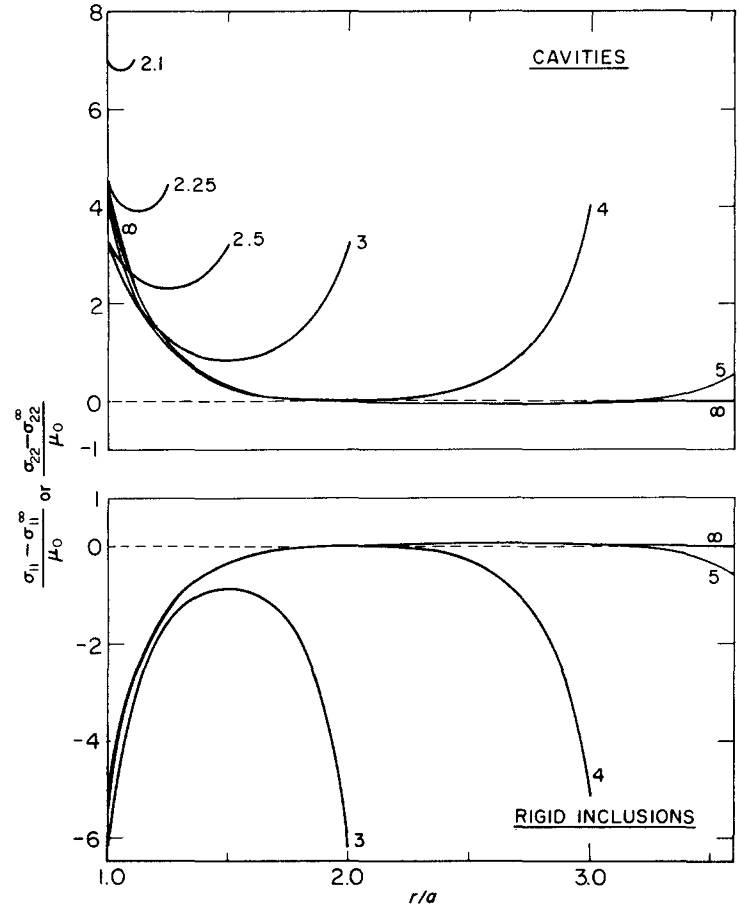

Fig. 5. Stresses along the centerline of the spheres for the applied strain $\epsilon^{\infty}_{ij} = \delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2} - 2\delta_{i3}\delta_{j3}$.

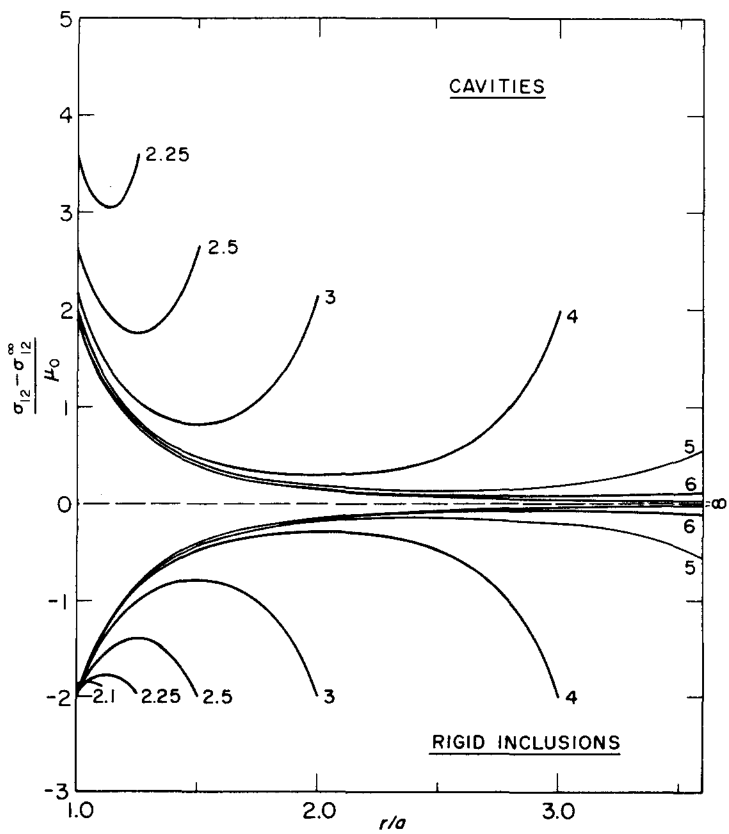

Fig. 6. Stresses along the centerline of the spheres for the applied strain $\epsilon^{\infty}_{ij} = \delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1}$.

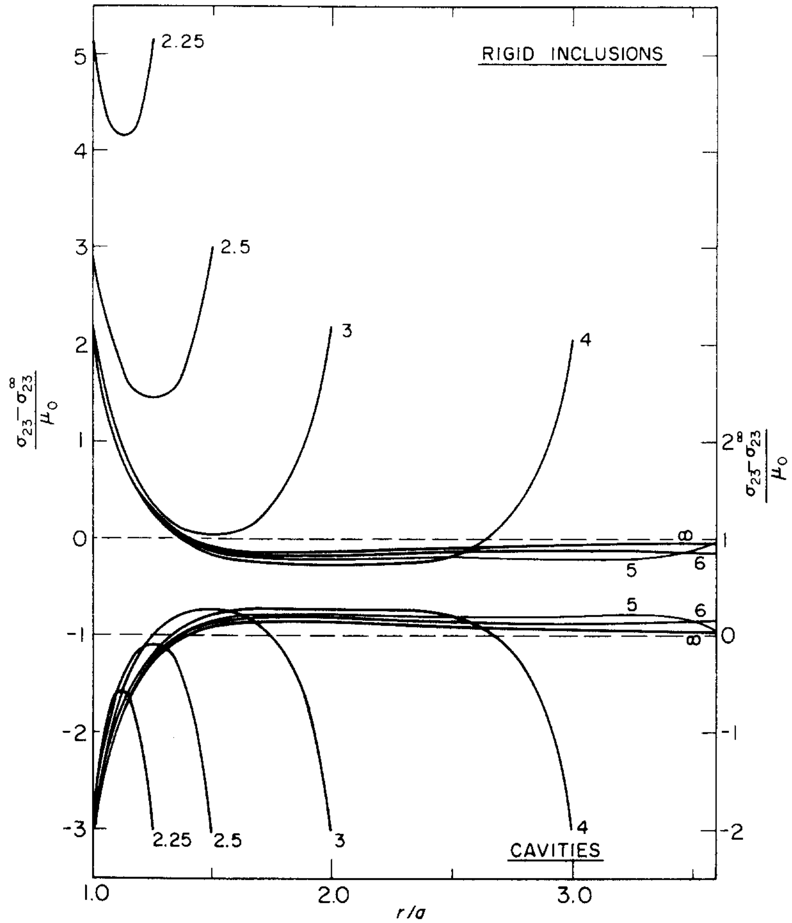

Fig. 7. Stresses along the centerline of the spheres for the applied strain $\epsilon_{ij}^{\infty} = \delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2}$.

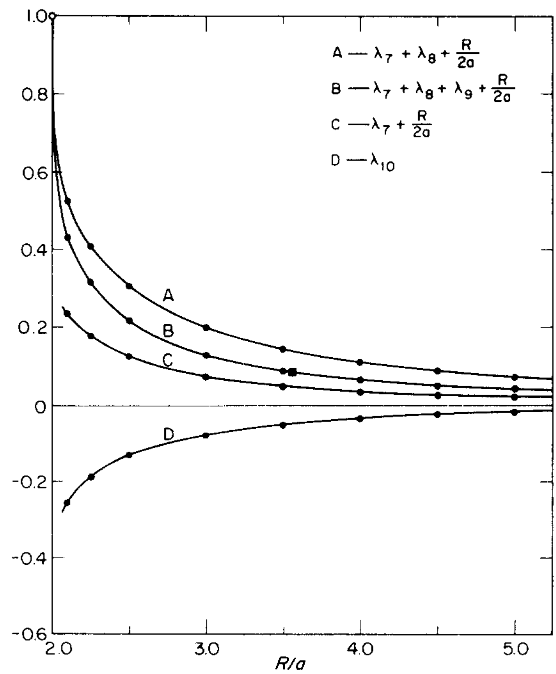

Fig. 8. The coefficients for the rigid particle displacement and rotation on sphere O. ●, Present work; ■, Shelley and Yu[2]; ○, Exact solution corresponding to zer displacement.

Table 1. Comparison of the stresses on the surface of cavity, at the point $\theta = \alpha = 0$, with the exact solution. Systems (a), (b) and (d) refer to the applied strains (1a), (1b) and (1d), respectively.

R/a	system (a) $\frac{\sigma_{33}-\sigma_{33}^{\infty}}{\mu_0}$ (n=40)	system (b) $\frac{\sigma_{33}-\sigma_{33}^{\infty}}{\mu_0}$ (n=70)	system (d) $\frac{\sigma_{23}-\sigma_{23}^{\infty}}{\mu_0}$ (n=50)
∞ (exact)	-5	4	-2
6	-4.99995	3.998	-2.00027
5	-4.9991	3.993	-2.00077
4	-4.9982	3.971	-2.0029
3	-4.972	3.777	-2.016
2.5	-4.833	3.132	-2.035
2.25	-4.556	2.284	-2.011
2.10	-3.626	1.485	-1.893

of accuracy with decreasing separation distance is clearly evident. Also for a given n, the accuracy seems to depend on the form of the applied strain and on the properties of the particles. Consequently, we conclude that the stresses shown in Figs. 2–7 are accurate only for $R/a \geq 3.0$. Also it should be noted that, in several cases, the stresses seem to have a singularity at $R/a = 2.0$, but owing to the loss of accuracy in the numerical calculations for $R/a \leq 3.0$, the precise form of the singularities cannot be inferred without further analysis and/or calculations. As shown in[6], however, the nature of the singularity for bulk quantities such as the stresslet S_{ij} can readily be determined via a "lubrication-type" expansion which becomes increasingly accurate as $(R/a)\to 2$.

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APPENDIX

The coefficients M_i are

$$M_1 = \frac{(1-\beta)(n-1)n(2n-1)}{2[\beta(n-1)(3n-4n\nu_0+2-2\nu_0)+(n^2+n+1-2n\nu_0-\nu_0)]}$$

$$M_2 = \frac{(1-\beta)(2n+5)(n+1)(n+5-4\nu_0)}{2[\beta(n+1)(3n-4n\nu_0+8-10\nu_0)+(n^2+5n+7-2n\nu_0-5\nu_0)]}$$

$$M_3 = \frac{n-4+4\nu_0}{2n-1}M_1$$

$$M_4 = \frac{n+1}{2n+1} M_1 + \frac{n-2+4\nu_0}{2n+3} M_2 - \frac{2[\beta(n^2+n+2n\nu_p+1+\nu_p)(3n-4n\nu_0+1-2\nu_0)-(3n-4n\nu_p+1-2\nu_p)(n^2+n+2n\nu_0+1+\nu_0)]}{(2n+1)(2n+3)[\beta(n^2+n+2n\nu_p+1+\nu_p)+(n+2)(3n-4n\nu_p+1-2\nu_p)]}$$

$$M_5 = \frac{n_1+3}{2n+5} M_2, \quad M_6 = \frac{(2n+3)(2n+1)n(1-\beta)}{2[\beta n(3n-4n\nu_0+5-6\nu_0)+(n^2+3n+3-2n\nu_0-3\nu_0)]}$$

$$M_7 = \frac{n-3+4\nu_0}{2n+1} M_6, \quad M_8 = \frac{n+2}{2n+3} M_6.$$

With the definition $A_n^* = A_n/(\mu_0\gamma_1)$ and $C_n^* = C_n/(\mu_0\gamma_1)$, eqns (14a, b) can be written as

$$A_n^* = 2\delta_{n0} + \sum_{s=0}^{\infty} A_s^* \rho^{s+n+1}(N_1 + N_2\rho^2) + \sum_{s=1}^{\infty} C_s^* \rho^{s+n}(N_3 + N_4\rho^2 + N_5\rho^4)$$

$$C_n^* = -\sum_{s=0}^{\infty} A_s^* \rho^{s+n+2} N_6 - \sum_{s=1}^{\infty} C_s^* \rho^{s+n+1}(N_7 + N_8\rho^2) \tag{A1}$$

where

$$N_1 = M_1\binom{s+n}{s} \qquad N_2 = M_2\binom{s+n+2}{s}$$

$$N_3 = M_1\binom{s+n}{s} - M_3\binom{s+n-1}{s} \qquad N_4 = M_2\binom{s+n+2}{s} - M_4\binom{s+n+2}{s}$$

$$N_5 = -M_5\binom{s+n+3}{s} \qquad N_6 = M_6\binom{s+n+1}{s}$$

$$N_7 = M_6\binom{s+n+1}{s} - M_7\binom{s+n}{s} \qquad N_8 = -M_8\binom{s+n+2}{s}.$$

Substituting the expansion $A_n^* = \sum_{m=0}^{\infty} A_{nm}\rho^m$ and $C_n^* = \sum_{m=0}^{\infty} C_{nm}\rho^m$ into eqn (A1) and equating the same powers of ρ yield

$$C_{n0} = 0$$

$$C_{nm} = -\sum_{s=0}^{m-n-2} N_6 A_{s(m-n-s-2)} - \sum_{s=1}^{m-n-1} N_7 C_{s(m-n-s-1)} - \sum_{s=1}^{m-n-3} N_8 C_{s(m-n-s-3)}$$

and

$$A_{n0} = 2\delta_{n0}$$

$$A_{nm} = \sum_{s=0}^{m-n-1} N_1 A_{sm-n-s-1)} + \sum_{s=0}^{m-n-3} N_2 A_{s(m-n-s-3)} + \sum_{s=1}^{m-n} N_3 C_{s(m-n-s)} + \sum_{s=1}^{m-n-2} N_4 C_{s(m-n-s-2)} + \sum_{s=1}^{m-n-4} N_5 C_{s(m-n-s-4)}.$$